2nd Microlocalization in the Gevrey classes : after S. Kishida(Functional-Analytic Study of Generalized Functions)

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2nd Microlocalization in the Gevrey classes (after S. Kishida)

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This talk aims at giving results due to S. Kishida, who was a promising student of H. Komatsu and became a man of another world in last April. His result is stimulating and will surely provide various further developments. I believe that it should be worthy to give even a rough description of his result in this talk.

§1. The result of J.M. Bony and P. Schapira.

In 1975, J.M. Bony and P. Schapira gave the following theorem concerning the propagation of singularities for a class of microdifferetnial equations . Let M be an open subset of R^n_x with a complex neighborhood X in C^n_z. Let Q be a microdifferential operator defined in a neighborhood of $\rho_0\in\mathring{T}^*_MX$. We assume the following conditions microlocally near ρ_0.

(1) $\Sigma=ch(Q)\cap\mathring{T}^*_MX$ is a regular involutory submanifld in $\mathring{T}^*_MX$ of codimension d.

Then we can write Σ as

$$\Sigma=\{\rho\in\mathring{T}^*_MX;\ p_1(\rho)=\cdots=p_d(\rho)=0\}$$

by real valued real analytic function $p_1,\cdot\cdot,p_d$ homogenous of order 1 satisfying

$$\{p_i,p_j\}\equiv 0 \qquad (i,j=1,\cdots,d).$$

Moreover we assume that $q=\sigma(Q)$ is written in the form

(2) $$q=\sum_{\alpha=|m|} q_\alpha p_1^{\alpha_1}\cdots p_d^{\alpha_d}$$

by homogeneous holomorphic functions q_α's and suppose that

(3) $$q_\Sigma=\sum_{|\alpha|=m} q_\alpha\Big|_\Sigma \tau_1^{\alpha_1}\cdots\tau_d^{\alpha_d}\neq 0$$

for any $\tau=(\tau_1,\cdots,\tau_d)\in R^d\setminus\{0\}$. Then we have

Theorem 1. (J.M. Bony and P. Schapira[B-S])

Let u be a microfunction solution to Pu=0. Then supp(u) is a union of d-dimensional bicharacteristic leaves of Σ.

This theorem can be easily recoverd by applyong the theory of 2nd microfunctions due to M. Kashiwara[K-L] and 2nd microdifferential operators due to Y. Laurent[Lr] in the following way.

By finding a suitable quantized contact transformation, we may assume $p_j=\xi_j$ where we take a coordinate of $\mathring{T}^*_M X$ as $(x,\xi dx)$ and that of T^*X as $(z,\zeta dz)$. Then p can be written in the form

$$q=\sum_{|\alpha|=d} q_\alpha(z,\zeta)\zeta'^\alpha$$

and the condtion (3) is equivalent to

(4) $$q_\Sigma=\sum_{|\alpha|=d} q_\alpha(x,\xi'=0,\xi'')x'^* \neq 0$$

for any $x'^*\in R^d\setminus\{0\}$. Here we set

$$\xi=(\xi',\xi'')=(\xi_1,\cdot\cdot,\xi_d;\xi_{d+1},\cdot\cdot,\xi_n),\quad \zeta=(\zeta',\zeta'')=(\zeta_1,\cdot\cdot,\zeta_d;\zeta_{d+1},\cdot\cdot,\zeta_n)$$

and take a coordinate of $T^*_{\tilde\Sigma}\tilde\Sigma$ as $(x,\xi''dx'';x'^*dx')$ with $x'^*=(x_1^*,\cdot\cdot,x_d^*)$, where $\tilde\Sigma$ denotes the union of all complex bicharacteristic leaves of Σ^C (a complexification of Σ in T^*X) passing through Σ. We can identify

$$\tilde\Sigma \xrightarrow{\sim} C^d_{z'}\times\sqrt{-1}T^*R^{n-d}_{(x'',\sqrt{-1}\xi''dx'')},$$

which is equipped with the sheaf $\mathscr{C}_{\tilde\Sigma}$ of microfunctions with holomorphic parameters z'. On $T^*_\Sigma\Sigma$, M. Kashiwara constructed the sheaf $\mathscr{C}^2_\Sigma$ of 2-microfunctions along Σ, by which we can study the properties microfunctions on Σ precisely. In fact, we have exact sequences $0 \longrightarrow \mathscr{C}_\Sigma|_\Sigma \longrightarrow \mathscr{B}^2_\Sigma \longrightarrow \mathring{\pi}_*\left(\mathscr{C}^2_\Sigma|_{\mathring{T}^*_\Sigma\Sigma}\right) \longrightarrow 0$

and

$$0 \longrightarrow \mathscr{C}_M|_\Sigma \longrightarrow \mathscr{B}^2_\Sigma.$$

Here we set $\mathscr{B}^2_\Sigma = \mathscr{C}^2_\Sigma|_\Sigma$.

On the other hand, Y. Laurent[Lr] constructed the theory of 2-microdifferntial operators along $\Sigma^{\mathbf{C}}$, which act on $\mathscr{C}^2_\Sigma$. First we regard $\Sigma^{\mathbf{C}}$ as a submanifold of $\Sigma^{\mathbf{C}}\times\Sigma^{\mathbf{C}}$ through the embeding

$$T^*X \simeq T^*_X(X\times X) \longrightarrow T^*(X\times X)$$

induced from $X \longrightarrow X\times X$. Then $\Sigma^{\mathbf{C}}$ is the union of all bicharacteristic leaves of $\Sigma^{\mathbf{C}}\times\Sigma^{\mathbf{C}}$ passing through $\Sigma^{\mathbf{C}}$ and then we can take a coordinate of $T^*_{\Sigma^{\mathbf{C}}}\Sigma^{\mathbf{C}}$ as $(z,\zeta''dz'',z'^*dz')$ with $z'^*=(z^*_1,\cdots,z^*_d)$.

On $T^*_{\Sigma^{\mathbf{C}}}\Sigma^{\mathbf{C}}$, Y. Laurent constructed the sheaf $\mathscr{E}^{2,\infty}_{\Sigma^{\mathbf{C}}}$ as follows. Take an open subset U of $T^*_{\Sigma^{\mathbf{C}}}\Sigma^{\mathbf{C}}$. Then the section $\mathscr{E}^{2,\infty}_{\Sigma^{\mathbf{C}}}(U)$ is the union of the formal sums $\sum_{i,j} P_{ij}(z,\zeta'',z'^*)$ satisfying the conditions (C1) and (C2) given below.

(C1) P_{ij} is holomorphic on U and homogeneous of order j with respect to (ζ'',z'^*) and of order i with respect to z'^*.

(C2) For any $\varepsilon>0$ and any compact subset K of U, there exists $C_{\varepsilon,K}$, and for any compact subset K of U there exists $C_K>0$ such that

$$\sup_K |P_{i,i+k}| \le \begin{cases} C_{\varepsilon,K}\varepsilon^{i+k}/i!k! & (i,k\ge 0) \\ C_{\varepsilon,k}^{-k}\ \varepsilon^{i}\ (-k)!/i! & (i\ge 0,\ k<0) \\ \varepsilon^{k}\ C_K^{-i}\ (-i)!/k! & (i<0,\ K\ge 0) \\ C_K^{-i-k}\ (-i)!(-k)! & (i,k<0). \end{cases}$$

Moreover Y. Laurent constructed the sheaf of 2-microdifferential operators of finite order as follows. $\sum_{ij} P_{ij}(z,\zeta'',\zeta'^*)\in\mathcal{E}^{2,\infty}_{\Sigma}\mathbf{C}$ belongs to $\mathcal{E}^2_{\Sigma}\mathbf{C}$ if and only there exists $j_0\in\mathbf{Z}$ such that

$$P_{ij}\equiv 0\ (j>j_0)$$

and for any $j\in\mathbf{Z}$, there exists $i(j)\in\mathbf{Z}$ such that

$$P_{ij}\equiv 0\quad (i<i(j)).$$

Then we can define the principal symbol for operatos of finite order. For $P=\sum_{i,j} P_{ij}(z,\zeta'',z'^*)\in\mathcal{E}^2_{\Sigma}\mathbf{C}$, we set

$$\sigma_{\Sigma}\mathbf{C}(P) = P_{i_0 j_0}$$

where

$$j_0=\max\{j;\ \text{for some } i,\ P_{ij}\not\equiv 0\},\ i_0=\min\{i;\ P_{ij}\not\equiv 0\}.$$

Then Y. Laurent proved that P is invertible in $\mathcal{E}^2_{\Sigma}\mathbf{C}$ if and only if $\sigma_{\Sigma}\mathbf{C}(P)$ does not vanishe. This fact implies that for any microfunction solution u to Qu=0, the condition (4) ensures that $u\in\mathcal{C}_{\Sigma}$. Then we can easily shows the assertion of the theorem.

Before entering into the result of Kishida, we remark that J.M. Bony[B] (see also G. Lebeau[Le]) and J. Sjöstrand[Sj1] showed a theorem concerning the propagation of C^∞ singularities under conditions on the lower terms. Moreover there are many results along this line. See Hanges-Sjöstrand[H-Sj], Kashiwara-Schapira[K-S] and A.

Kessab[Ke].

§2. Kishida's result

Hereafter we follow the notation prepared in the latter part of §1 and we restrict ourselves to the case of the canoncial form:

$$\Sigma=\{(x,\sqrt{-1}\xi\cdot dx);\ \xi_1=\cdots=\xi_d=0\}.$$

Let P be a differential operators of order m defined in a neighborhood of $\mathring{x}\in M$. We assume the condition (1) and that $p=\sigma(P)$ vanishes strictly of order ℓ: p is written as vanishes strictly of order ℓ: p is written as

$$p=\sum_{|\alpha|=\ell} p_\alpha \xi'^\alpha$$

by homgeneous holomorphic functions p_α's and

(2)' $$p_\Sigma=\sum_{|\alpha|=\ell} p_\alpha(x,\xi'=0,\xi'')x'^{*\alpha}\neq 0$$

for any $x'^*\neq 0$ and $(x,\xi''dx'')$ near $(\mathring{x},\sqrt{-1}dx_n)$. Moreover we assume a condition on the lower term as follows. We develop $p_j(z,\xi)$ into the partial Taylor series with respect to ξ' as

$$P_j=\sum_\alpha P_j^\alpha \xi_1^{\alpha_1}\cdots\xi_d^{\alpha_d}$$

and set

$$P_{ij}=\sum_{|\alpha|=i} P_j^\alpha \xi_1^{\alpha_1}\cdots\xi_d^{\alpha_d}.$$

To define a invariant for P, we set

$$i(j)=\inf\{i;\ P_{ij}\equiv 0\},$$

by which we can deine the Newton polygon for P along $\Sigma^{\mathbf{C}}$. Moreover we set

$$\sigma=\max\{(i(m)-i(j))/(m-j),\ 1\}$$

and call it the irregularity of P along $\Sigma^{\mathbf{C}}$. Hence we give

Theorem 2. (Kishida[Ki])

Let $u\in\mathcal{D}\mathcal{E}_{\mathring{x}}$ *satisfying* $Pu\in\mathcal{A}$. *Then* $WF^{(s)}(u)$ *is a union of bicharacteristic leaves of* Σ *in a neighborhood of* ρ_0 *for* $s\le\sigma/(\sigma-1)$.

Here we give the definition of the Gevrey wave front sets by using FBI transformation. We identify T^*R^n with C^n as

$$\chi:\quad (x,\xi)\longmapsto z=x-\sqrt{-1}\xi.$$

Then we set for $f\in\mathcal{E}'(R^n)$

$$(T^1f)(z,\lambda)=\int\exp(-\lambda(z-x)^2/2)f(x)dx.$$

Then $\mathring{\rho}=(\mathring{x},\mathring{\xi})\in WF^{(s)}(f)$ if and only if there exists a neighborhood ω of $\chi(\mathring{\rho})$ and $\lambda_0\ge1$, and there exists $C_\varepsilon>0$ for any $\varepsilon>0$ such that

$$|T^1f(z,\lambda)|\le\exp(\lambda|\mathrm{Im}\ z|/2)\cdot e^{-\varepsilon\lambda^{1/s}}\quad(\lambda\ge\lambda_0).$$

§3. Proof of Theorem 2.

First of all, we define the notion of Gevrey 2-microsupports as follows. We take a coordinate of $T_\Sigma T^*R^n$ as $(x,\xi''dx'';x'^*\partial/\partial\xi')$ and identify $T_\Sigma T^*R^n$ with C^n as

$$\chi_\Sigma:\quad (x,\xi'',x'^*)\longmapsto(x'-\sqrt{-1}x'^*,x''-\sqrt{-1}\xi'').$$

We put $u(z,\lambda)=T^1f(z,\lambda)$ for $f\in\mathcal{E}'(R^n)$ and set

$$T_\Sigma^2u(z,\lambda,\mu)=\int_{|x'-\mathrm{Re}\ \mathring{z}'|\le\delta}\exp\left(-\frac{\mu\lambda}{2(1-\mu)}(z'-x')^2\right)u(x',z'')dx'.$$

Then for s satisfying $1<s<+\infty$, $\mathring{\tau}=(\mathring{x},\mathring{\xi}'',\mathring{x}'^*)\in WF_\Sigma^{2,s}(f)$ if and only if there exist positive numbers ε,r,μ_0 and Λ_0 sstisfying

$$|T_\Sigma^2u(z,\lambda,\mu)|\le\exp\left(\frac{\lambda\mu}{2}|\mathrm{Im}\ z'|^2+\frac{\lambda}{2}|\mathrm{Im}\ z''|^2-\varepsilon\mu\lambda\right)$$

on $\{0<\mu<\mu_0,\ \lambda>f(\mu):=\Lambda_0\mu^{-s/(s-1)},\ |z-\chi(\mathring{\tau})|<r\}$.

We give a definition of a function space which is the image of 2nd FBI transformation.

Definition. Let ω be an open subset of C^n. Then a function

$u(z,\mu,\lambda)$ defined on $\{z\in\omega,\mu>0,\lambda\geq 1\}$ belongs to $H^{2,\sigma}_{\Sigma}$ if

(i) u is holomorphic with respect to z.

(ii) For any compact set K in ω and $\varepsilon>0$, there exist $\mu_0,c_0,C_0>0$ satisfying

$$|u(z,\mu,\lambda)|\leq C_0 e^{\{\mu\lambda(\mathrm{Im}\ z')^2/2\}+\{\lambda(\mathrm{Im}\ z'')^2/2\}}$$
$$\text{on } \{z\in K,\ 0<\mu<\mu_0,\ c_0\lambda^{1/\sigma}\leq\lambda\mu\}.$$

We regard $u(z,\mu,\lambda)$ as 0 at $\mathring{z}\in\omega$ if and only if there exist an open neighborhood U of $\mathring{z}$ and $\delta,\mu_0,c_0,C_0>0$ such that

$$|u(z,\mu,\lambda)|\leq C_0 e^{\{\mu\lambda(\mathrm{Im}\ z')^2/2\}+\{\lambda(\mathrm{Im}\ z'')^2/2\}-\delta\lambda\mu}$$
$$\text{on } \{z\in U,\ 0<\mu<\mu_0,\ c_0\lambda^{1/\sigma}\leq\lambda\mu\}.$$

Important is the fact that

$$\pi_{\Sigma}(WF^{2,(s)}_{\Sigma}(f))\subset WF^{(s)}(f)\qquad(\pi_{\Sigma}:T_{\Sigma}T^{*}R^{n}\longrightarrow\Sigma).$$

Moreover outside of $\pi_{\Sigma}(WF^{2,(s)}_{\Sigma}(f))$, $WF^{(s)}(f)$ has the unique continuation property along the bicharacteristic leaves of Σ. Thus it suffices to show that $WF^{2,(s)}_{\Sigma}(f)=\phi$ for any $f\in\mathcal{D}\mathcal{b}$ satisfying $Pf\in\mathcal{A}$. This fact is shown by employing 2-pseudodifferential operators acting on T^2T^1f as follows.

First we recall the notion of 2-analytic symbols due to J. Sjöstrand and G. Lebeau[Le].

Definition. Let Ω be an open subset of $\mathbf{C}^n$ and φ a continuous function $\Omega\longrightarrow R_+$. Then a function $u(z,\mu,\lambda)$ defined on $\{z\in\Omega,\ \mu>0,\ \lambda\geq 1\}$ belongs to $H^2_{\varphi}(\Omega)$ if and only if the conditions (i) and (ii) are satisfied.

(i) u is holomorphic with respect to z.

(ii) For any $\varepsilon>0$ and any compact set K of Ω, there exist $\mu_0, C_0>0$ and a decreasing function $f(\mu)$: $]0,\mu_0[\longrightarrow \mathbb{R}$ such that

$$|u(z,\mu,\lambda)| \leq C_0 e^{\mu\lambda(\varphi(z)+\varepsilon)} \quad (z\in K,\ 0<\mu<\mu_0,\ \lambda>f(\mu)).$$

Element of $H^2_\varphi(\Omega)$ is called 2-analytic symbols.

Moreover we define a subclass of 2-analytic symbols by posing a additional condition on the grouth of symbols.

Definition. A 2-analytic symbol $u(z,\mu,\lambda)$ is called σ-tempered if for any compact set $K\subset\Omega$ and any $\varepsilon>0$, there exist $\mu_0, c_0, C_0>0$ such that

$$(ii)_\sigma \quad |u(z,\mu,\lambda)| \leq C_0 e^{\mu\lambda(\varphi(z)+\varepsilon)} \quad (z\in K,\ 0<\mu<\mu_0,\ c_0\leq\lambda\mu\leq\lambda\mu_0).$$

Moreover σ-temperd 2-analytic symbol $u(z,\mu,\lambda)$ is of finite order if and only if there exist $N>0$ and $C_0, c_0>0$ such that

$$|u(z,\mu,\lambda)| \leq \tilde{\lambda}^N \quad (c\lambda^{1/\sigma}\leq\lambda\mu\leq\lambda\mu_0).$$

To study the invertibility of operators 2-microlocally, we introduce the formal symblos following Y. Laurent[Lr].

Definition. A formal sum $p(z,\mu,\lambda)\sim\sum p_{k,k+i}(z)\tilde{\lambda}^k\mu^i$ $(\tilde{\lambda}=\lambda\mu)$ is called a formal 2-analytic symbol on Ω of growth $(r,1)$ if and only if the following conditions (i) and (ii) are satisfied.

(i) $p_{k,k+i}(z)$ is holomorphic on Ω.

(ii) For any compact subset K in Ω, there exists a positive C_K and for any compact set K in Ω and any $\varepsilon>0$ such that

$$\sup_K |P_{i,i+k}| \le \begin{cases} C_\varepsilon^k\ \varepsilon^i\ /i!(k!)^r & (i,k\ge 0) \\ C_\varepsilon^{-k}\ \varepsilon^i\ (-k)!/i! & (i\ge 0,\ k<0) \\ C_K^k\ C_K^{-i}\ (-i)!/(k!)^r & (i<0,\ K\ge 0) \\ C_K^{-i-k}\ (-i)!(-k)! & (i,k<0). \end{cases}$$

Moreover we define formal symbols of finite order as follows.

Definition. A formal 2-analytic symbols of growth $(r,1)$ $\sum p_{i,i+k}$ is called of type $(r,1)$ of bidegree $[i_0,j_0]$ denoted by $\in E^2(r,1)[i_0,j_0]$ if and only if

$p_{i,i+k}\not\equiv 0$ implies $(i/r)+k\le(i_0/r)+(j_0-k_0)$ and $i+k\le j_0$.

We set $E^2(r,1)=\bigcup E^2(r,1)[i_0,j_0]$. For $p(z,\mu,\lambda)\in E^2(r,1)$, we put

$$\sigma^2(p)(z,\mu,\lambda)=p_{i_0,j_0}\ \tilde{\lambda}^k\mu^i$$

called prinsipal part of p, where p does not belong to $E^2(r,1)[i_0,j_0]$ which is strictly smaller than $E^2_{(r,1)}[i_0,j_0]$.

We remark that formal 2-analytic symbols of growth $(r,1)$ define σ-tempered 2-analytic symbols in case $\sigma<r$.

Hereafter Ω denotes an open subset of $C^{2n}_{(z,\xi)}$ and we define a action of σ-temperd 2-analytic symbols of finite order on $H^{2,\sigma}_\Sigma$ as

$$(:p:u)(z,\mu,\lambda)$$
$$=(\frac{\lambda}{2\pi})^{n-d}(\frac{\lambda\mu}{2\pi})^d\cdot\int e^{\sqrt{-1}\mu\lambda(x'-y')\theta'+\sqrt{-1}\lambda(x''-y'')\theta''}$$
$$p(x,\theta,\mu,\lambda)u(y,\mu,\lambda)dyd\theta.$$

Then crucial is the fact that the invertibility of the operator :p: is determined by the principal symbol of p if p is formal symbol. In fact we have

Proposition 3.1. *For* $p \in E^2(r,1)$, $:p:$ *is invertible if* $\sigma^2(p)$ *does not vanish.*

in Y. Laurent[Lr] for $\mathcal{E}^2_\Lambda(r,1)$.

The above proposition can be proved essentially the same way as in Y. Laurent [Lr] for $\mathcal{E}^2_\Lambda$.

We go back to the proof of Theorem 2. By the above proposition, we can easily prove $WF^{2,(s)}_\Sigma(f)=\phi$ for $f \in \mathcal{D}\mathcal{b}$ satisfying $Pf \in \mathcal{A}$.

Remark. In the analytic category Lubin-Esser [LbE] presented a theory of 2-pseudodifferential operators based on FBI transformation. Moreover P. Esser[E] also gave a definition of 2nd wavefronts for distributions. But we can find some differences between the two definitions.

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